Stochastic resonance in noisy maps as dynamical threshold-crossing systems

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Interplay of noise and periodic modulation of system parameters for the logistic map in the region after the first bifurcation and for the kicked spin model with Ising anisotropy and damping is considered. For both maps two distinct symmetric states are present that correspond to different phases of the period-2 orbit of the logistic map and to disjoint attractors of the spin map. The periodic force modulates the transition probabilities from any state to the opposite one symmetrically. It follows that the maps behave as threshold-crossing systems with internal dynamics, and stochastic resonance (maximum of the signal-to-noise ratio in the signal reflecting the occurrence of jumps between the symmetric states) in both models is observed. Numerical simulations agree qualitatively with analytic results based on the adiabatic theory.

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I. INTRODUCTION

Stochastic resonance (SR) is a nonlinear phenomenon in which the transmission of a coherent signal by certain systems can be improved by the addition of noise to the system [1]. The essence of the phenomenon is that even a weak periodic signal which can be undetectable in the absence of noise can be detected in the presence of optimum noise, e.g., in a bistable system one can observe a strong periodic component in the process of switching between two states [2]. Then the output power spectrum density (PSD) $S(f)$ will consist of peaks located at the multiples of the periodic signal frequency f_0, superimposed on the noise background $S_N(f)$. As a measure of SR the signal-to-noise ratio (SNR) is often used, defined as $\mathrm{SNR}=S_P(f_0)/S_N(f_0)$, where $S_P(f_0)=S(f_0)-S_N(f_0)$ is the first peak height. SR was observed experimentally and predicted theoretically in a large variety of systems, including optical systems [3,4], sensory neurons [5,6], signal transmission in ion channels [7], chaotic systems [8–10], electronic circuits [11], etc., to list only a few (for recent review see Ref. [12]).

Although first observations of SR were obtained for dynamical systems with bistable potentials [1–3], recently there has been a growing interest in the investigation of SR in dynamical and nondynamical threshold-crossing (TC) systems [5,6,13–18]. TC systems are often referred to as excitable systems [5,6] because their outputs consist of pulses that can be emitted each time the noisy input crosses some threshold. The important feature of SR in TC systems is a different kind of signals that are observed as compared to ‘‘conventional’’ SR in bistable systems. In fact, noise in TC systems is optimized to increase the periodicity of such events as pulses or jumps *between* accessible states once per period of the periodic part of the input but not the periodicity of *occupation* of such states. It follows that TC systems include both monostable and bistable models; in the latter case they can exhibit SR even if the conventional SR cannot be observed.

In the present paper we demonstrate that SR is possible in *discrete-time dynamical TC systems*. We investigate two maps that are subjected to the action of periodic input and noise. Such systems are easy to simulate numerically [9,19–22] and in many cases they retain essential features of SR in continuous-time stochastic systems. A detailed study of SR in bistable maps and in coupled map lattices was performed in [21], and then extended [22] to compare with a model of spatiotemporal SR based on the ϕ^4 theory [23]. Chaotic maps are basic models for the investigation of noise-free SR in which the internal chaotic dynamics is used to optimize SNR without the use of external noise [9,10,15,16]. However, with an exception of some artificially constructed models in [15,16], all maps analyzed so far in the context of SR were discrete-time simplifications of a generic dynamical continuous-time system with conventional SR: a model of an overdamped particle in a bistable potential. In such a model, additive periodic forcing changes alternately the *relative depth* of the potential wells, increasing the probability of jumps between wells twice per modulation period, and the probability of occupation of every well once per period [2]. As far as we know none of the previous studies of SR in maps (except of [15,16]) considered a TC system.

First the logistic map close to a bifurcation point from the period-1 to the period-2 state is analyzed in the presence of additive periodic forcing and noise. It is shown that in systems which exhibit period doubling SR can be realized by choosing the noise intensity to maximize the periodicity of phase jumps which reverse the order of points on the period-2 orbit. This study is in line with other studies of SR in systems close to period-doubling bifurcation [24], and it is in a sense complementary to the studies of SR in other routes to chaos, e.g., via intermittency [10]. Second we consider the spin map which is a model for the dynamics of a damped spin in the presence of anisotropy, driven by periodic pulses of magnetic field [25–29]. If the periodic and noisy components are added to the amplitude of the pulses, the noise intensity can be chosen in such a way that the jumps between

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two equivalent spin orientations occur most probably when the periodic component is at a maximum. In both examples investigations based on numerical simulations are compared with predictions of simple adiabatic theories.

One needs to stress that although both our systems possess two equivalent stable states their existence plays a quite different role as compared to conventional bistable models of SR studied first in [1–3] and then in many other papers. In such bistable models the external periodic force in fact leads to *the symmetry breaking* between both states (e.g., energy wells), i.e., to temporary ''energy'' increase of one well and energy decrease of the other well. It follows that there are time moments when the probability of the noise-induced jump, e.g., from the left to the right well, is larger than that of the reverse jump, and probabilities of occupations of both wells vary periodically in time. However, in both systems studied in the present paper, the external periodic force *does not break the system symmetry* (relative energies of both states are the same). This means that in an equivalent bistable model just the height of the barrier between both wells oscillates in time. As a result, at the same time moments one observes the increase of the jump probability from the left to the right well *and* that of the reverse jump. It follows that the probability of occupation of the wells is not modulated and the conventional SR cannot be observed in our models. Similar models with continuous time were investigated in [30,31] and they can be described in a general framework of theory of SR in TC systems, where the only manifestation of bistability is that the TC rate is that of surmounting the potential barrier.

II. SYSTEMS UNDER STUDY

A. The logistic map

Let us consider the logistic map

$$x_{n+1}=f(x_n)=rx_n(1-x_n) \tag{1}$$

with the control parameter r. For $1<r<3$ this map has one stable fixed point $x^*=1-1/r$ which loses stability in a period-doubling bifurcation at $r=r_1=3$. For $r>3$ a period-2 orbit consisting of points $x^{(1)}<x^*$ and $x^{(2)}>x^*$ occurs. These points are stable fixed points of the map f^2, i.e., $x^{(i)}=f^2(x^{(i)})$, $i=1,2$, while x^* is an unstable fixed point of f^2 being a separatrix between the basins of attraction of $x^{(1)}$, $x^{(2)}$. We consider the logistic map with $r>r_1$, driven by additive periodic forcing and noise

$$x_{n+1}=rx_n(1-x_n)+A_0\cos(\omega_0 n)+D\eta(n), \tag{2}$$

where $\eta(n)$ is a zero-mean white Gaussian noise with variance one. Henceforth we assume that the modulation period $2\pi/\omega_0$ is even. We take care that x_n does not leave the interval $(0;1)$ by a slight modification of the Gaussian distribution so that $\eta(n)$ cannot assume values larger than a given cutoff; however, since we deal with $D\ll 1$ (cf. Sec. III A) this modification is very small.

If the amplitude A_0 is below a certain threshold, then in the absence of noise the time series of x_n from Eq. (2) still shows an overall pattern of the period-2 orbit, i.e., x_n is alternately below and above the separatrix x^*. The location of points visited by the trajectory is periodically modulated with frequency ω_0 [Fig. 1(a)]. We assume that $\omega_0\ll 1$ and it follows that for any given n the points of the trajectory and the separatrix are close to three fixed points of the second iteration of Eq. (2) with $D=0$

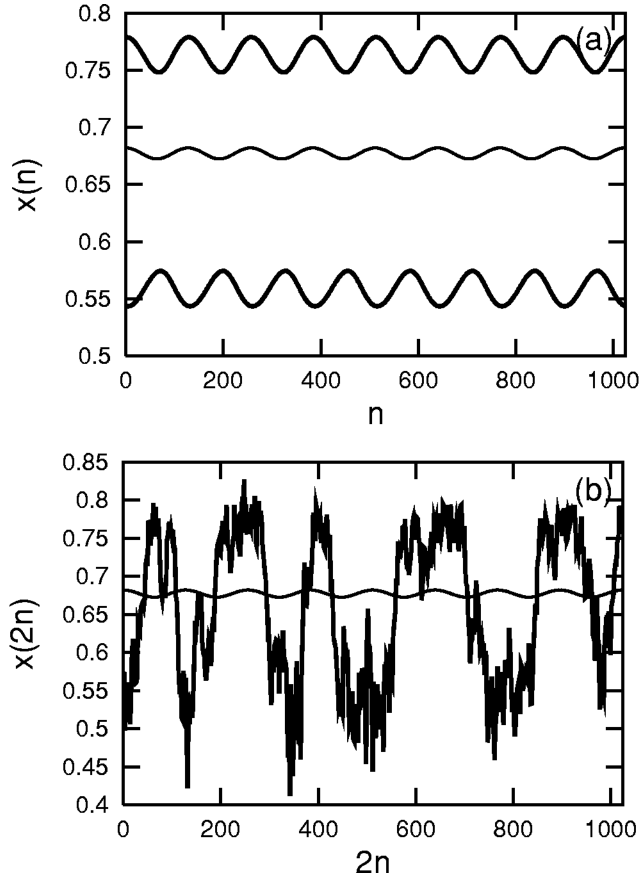

FIG. 1. (a) Fixed points of the second iteration of the logistic map with periodic forcing and without noise [Eq. (2) with $r=3.1$, $A_0=0.01$, $2\pi/\omega_0=128$]: stable $x^{(1)}(n)$, $x^{(2)}(n)$ [thick lines, from the numerical simulation of Eq. (2)] and unstable separatrix $x^{(*)}(n)$ [thin line, from Eq. (3)]; (b) every second iteration of Eq. (2) with $D=0.02$ and other parameters as in (a).

$$x^{(i)}(n)=f(f(x^{(i)}(n))+A_0\cos\omega_0 n)+A_0\cos\omega_0 n, \tag{3}$$

where f denotes the map (1) and $i=*,1,2$. In Fig. 1(a) the location of the separatrix $x^*(n)$ calculated from Eq. (3) is also shown. If only every second iteration of Eq. (2) is observed the points x_n perform periodic oscillations with frequency ω_0 always either above x^* or below x^*, depending on the initial condition.

It can be seen from Fig. 1(a) that the points below and above $x^*(n)$ are both close to or far from the separatrix at the same moments, in phase with the $\cos\omega_0 n$ forcing. We are interested in the perturbations of the overall period-2 pattern discussed above by the cooperative influence of the periodic forcing and noise. The periodic forcing is assumed to be too small to induce phase jumps in the overall period-2 pattern of x_n, i.e., to cause that if $x_n<x^*(n)$ then also $x_{n+1}<x^*(n+1)$ or if $x_n>x^*(n)$ then also $x_{n+1}>x^*(n+1)$ [cf. Fig. 1(a)]. However, with $D\neq 0$ such phase jumps are possible: e.g., if $x_n<x^*(n)$ then after applying the deterministic part of the map (2) the point will be above the separatrix

$x^*(n+1)$, but the addition of Gaussian noise can shift it back below the separatrix. If only every second iteration of Eq. (2) is observed, switching between small oscillations of x_n below and above the separatrix can be seen [Fig. 1(b)]. Assuming that after applying the deterministic part of Eq. (2) to any $x_n < x^*(n)$ the image of this point will be close to $x^{(2)}(n+1)$ and vice versa, it can be seen that maximum probability of the occurrence of such phase jumps corresponds to the maxima of the periodic component of Eq. (2), when both stable fixed points of Eq. (3) $x^{(1)}(n)$, $x^{(2)}(n)$ are close to the separatrix $x^*(n)$. This happens once per period $2\pi/\omega_0$. This situation resembles that in dynamical TC systems, where the subthreshold periodic forcing modulates the probability of noise-induced jumps over a threshold. In our case the function $f(x_n)$ in Eq. (2) represents the internal period-2 deterministic dynamics, and the noise-induced phase jumps are just shifts of the points x_n below or above the separatrix which plays a role of the threshold. Hence the occurrence of SR with varying noise intensity D can be expected if the output signal which reflects the occurrence or absence of *the phase jump of the period-2 orbit* (not the jumps between the states below and above the separatrix) at every time step n is analyzed.

In order to simplify the theoretical description of the above-mentioned system (Sec. III B) we also modified it as follows. First, for $D=0$ and given A_0 and ω_0 the location of points visited by the trajectory as in Fig. 1(a) is found from the numerical simulation of Eq. (2) and stored. Thus we have two series of points, $\tilde{x}^{(1)}(n) < x^*(n)$ and $\tilde{x}^{(2)}(n) > x^*(n)$, both of which are periodic. The points visited are not necessarily equal to $x^{(1,2)}$ from Eq. (3), in particular for high ω_0. Then, in the course of simulation of Eq. (2) with $D \neq 0$, $\tilde{x}^{(1)}(n)$ or $\tilde{x}^{(2)}(n)$ is set as the initial condition for the iteration $n+1$, if after the iteration n we have $x_n < x^*(n)$ or $x_n > x^*(n)$, respectively [since $\tilde{x}^{(1,2)}(n)$ are periodic, n on the rhs of these equations can be taken as mod $2\pi/\omega_0$]. We call this system the logistic map with reset. In this case the mechanism of noise-induced phase jumps is the same as discussed previously and SR can also be expected if the output signal reflects the occurrence or absence of the phase jump of the period-2 orbit at time n.

B. The model of dynamics of a kicked damped spin

In Refs. [25–29] a classical spin $\mathbf{S}$, $|\mathbf{S}|=S$ in the uniaxial anisotropy field with transverse magnetic field $\tilde{B}(t)$ added along the x axis was studied. The system is described by the Hamiltonian

$$H=-A(S_z)^2-\tilde{B}(t)S_x, \tag{4}$$

where A is the anisotropy constant. This classical model is related to experimentally investigated quantum magnetic systems if one considers the properties of isolated spins of large magnetic molecules such as $Mn_{12}O_{12}(CH_3COO)_{16}$, where the anisotropy is induced by molecule symmetry [32] or the nanometric-size single-domain ferromagnetic particles (superparamagnets) used for the observation of the macroscopic quantum tunneling phenomenon [33]. The motion of the spin is determined by the Landau-Lifschitz equation with the damping term

$$\frac{d\mathbf{S}}{dt}=\mathbf{S}\times\mathbf{B}_{eff}-\frac{\lambda}{S}\mathbf{S}\times(\mathbf{S}\times\mathbf{B}_{eff}), \tag{5}$$

where $\mathbf{B}_{eff}=-dH/d\mathbf{S}$ is the effective magnetic field and $\lambda>0$ is the damping parameter. Taking the transverse field in the form of periodic δ pulses with the amplitude B and the period τ

$$\tilde{B}(t)=B\sum_{n=1}^{\infty}\delta(t-n\tau) \tag{6}$$

and profiting from the fact that S is constant, Eq. (5) can be transformed into a superposition of two two-dimensional (2D) maps T_A and T_B. The map T_A describes the time evolution between kicks

$$T_A\begin{bmatrix}\varphi\\ S_z\end{bmatrix}=\begin{bmatrix}\varphi+\Delta\varphi\\ WS_z\end{bmatrix}, \tag{7}$$

where φ is the angle between the x axis and the projection of the spin on the x,y plane and

$$W=[c^2+(S_z/S)^2(1-c^2)]^{-1/2},$$

$$c=\exp(-2\lambda AS\tau),$$

$$\Delta\varphi=(1/\lambda)\ln\{(1+S/S_z)[1+S/(WS_z)]\}-2AS\tau.$$

The map T_B written in the variables (S_x,Φ), where Φ is the angle between the y axis and the projection of the spin on the x,z plane has a form

$$T_B\begin{bmatrix}\Phi\\ S_x\end{bmatrix}=\begin{bmatrix}\Phi-B\\ S-2S(S-S_x)D^2U\end{bmatrix}, \tag{8}$$

where $D=\exp(-\lambda B)$ and $U=[S+S_x+D^2(S-S_x)]^{-1}$. The complete dynamics is a superposition of the two maps

$$\mathbf{S}_{n+1}=T_B[T_A[\mathbf{S}_n]]. \tag{9}$$

We take B as the control parameter. For $S=1$, $\tau=2\pi$, $\lambda=0.105\,494\,2$, $A=1$, and $B<B_c=1$ two symmetric attractors of Eq. (9) exist [25,26]. These attractors correspond to two Ising states (spin ''up'' and ''down'') existing in the absence of the external field, and in general they can be fixed points, periodic orbits or chaotic attractors, depending on B [Fig. 2(a)]. For $B>B_c$ two symmetric strange attractors, which exist in the system when B is slightly below B_c, merge as a result of a crisis [27–29,34] [Fig. 2(a)]. We investigated Eq. (9) with the time-dependent control parameter

$$B(n)=B_0+B_1\cos(\omega_0 n)+D\eta(n) \tag{10}$$

with $B_0+B_1<B_c$. The values of B_0 and B_1 were chosen so that for $B=$const and $B_0-B_1<B<B_0+B_1$ the attractors of Eq. (9) are two stable fixed points. For $D=0$ the simulation of Eq. (9) with $B=B(n)$ given by Eq. (10) reveals that the spin is always up, $S_z>0$, or down, $S_z<0$, depending on initial conditions. In the presence of noise these two attractors merge into one due to the occurrence of the noise-induced attractor merging crisis [Fig. 2(b)] [35,36]. Neglecting possible transients one can say that switching between

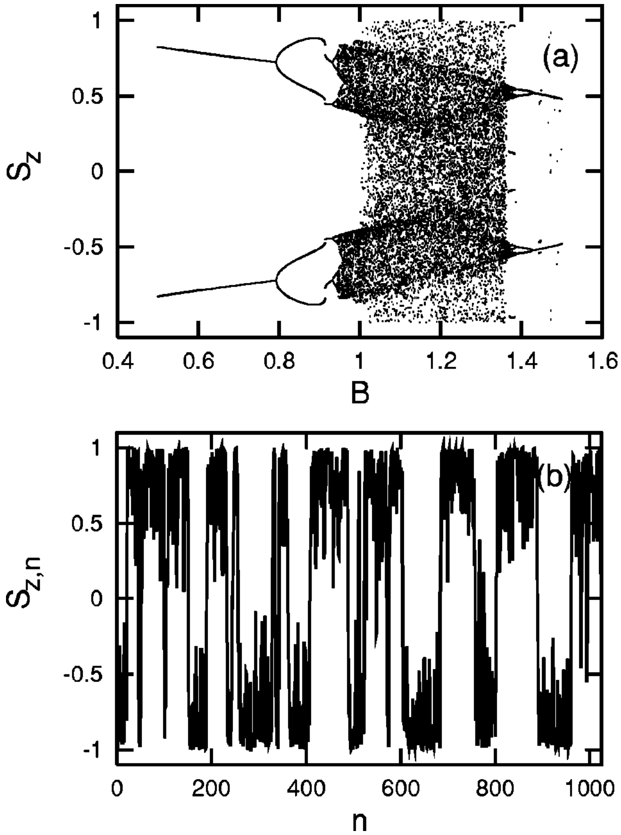

FIG. 2. (a) Bifurcation diagram S_z vs B for the spin map (9) with $\tau=2\pi$, $\lambda=0.105\,494\,2$, $A=1$; (b) time series of $S_{z,n}$ from Eq. (9) with $B_0=0.5$, $B_1=0.1$, $D=0.55$, $2\pi/\omega_0=128$, and other parameters as in (a).

the two parts of the attractor can take place when in Eq. (10) $B>B_c$. Jumps between the two parts of the attractor are most probable when the periodic forcing is at a maximum, once per period $2\pi/\omega_0$. Hence SR will occur for the optimum value of D if the signal analyzed reflects the occurrence or absence of the jump (see Sec. III B for details). The difference with the logistic map is that here the periodic forcing and noise are multiplicative and their effect is dynamical: they modify the control parameter, and, what follows, the internal system dynamics, instead of shifting the points directly below or above the separatrix. The complicated internal dynamics of the map (9) causes that the jumps need not necessarily occur always when $B(n)>B_c$.

At this point the difference between our systems and the bistable 1D maps studied in [9,19–22] should be further emphasized. Both maps (2) and (9) are bistable. In the logistic map the two symmetric states are two period-2 orbits with a reverse order of points on the orbit. In the spin map with $B<B_c$ two symmetric disjoint attractors exist. When SR is studied in generic bistable systems, the model of the particle in the potential with two symmetric wells, left and right, is usually applicable. The additive periodic force modulates the relative depth of the potential wells in such a way that during half a period, e.g., the left well is deeper, and during the next half—the right one [2]. Thus the probabilities (or rates) of transitions from the wells are also modulated, and there is a phase shift, equal to half of the modulation period, between the probabilities of transitions from the left and right well. Also the probabilities of occupation of the two wells are modulated with the same relative phase shift. SR at frequency $f_0=\omega_0/2\pi$ is observed if a signal reflecting the *actual position* of the particle in the left or right well is analyzed.

In this paper, in both systems under study, the periodic force modulates the transition probabilities from the two above-mentioned states *symmetrically*, without the relative phase shift, and thus the probability of occupation of any of these states *is not* modulated. However, the probability of jumps between the two states is modulated with period $2\pi/\omega_0$. SR at frequency $f_0=\omega_0/2\pi$ can thus be expected if the signal reflecting the presence or absence of *the jump between the system states* at a given moment is analyzed. As pointed out in Sec. I, this is a typical situation in the model of the particle in the bistable potential with modulated barrier height. In our models the internal dynamics within the two states is also taken into account. Thus the bistability of our systems plays a quite different role as compared to conventional observations of SR [2] and the models under study bear a certain resemblance to *dynamical TC systems*.

III. STOCHASTIC RESONANCE

A. Numerical results and discussion

In this section we present the numerical results for SNR vs D obtained in the systems discussed in Sec. II. In all cases we analyzed a two-state signal: $y_n=1$ if at time step n the phase jump in Eq. (2) or the jump between symmetric parts of the attractor in Eq. (9) occurred and $y_n=0$ otherwise. The phase jump at time step n in the logistic map model occurs if x_n is below (above) the separatrix $x^*(n)$, though x_{n-1} was also below (above) the separatrix, respectively. The location of the separatrix for every n is evaluated from Eq. (3). We assume that the jump between symmetric parts of the attractor in the spin map model occurs when $S_{z,n}$ and $S_{z,n-1}$ have opposite signs. Thus $S_z=0$ is assumed as a separatrix between the attractors. This is not strict since the structure of the basin boundaries of chaotic attractors in Eq. (9) is very complicated [27–29], but such an assumption is enough for our purposes. In all cases the PSD was calculated from 32 678 points of the signal y_n using fast Fourier transform (FFT) with a square window, and SNR was evaluated as defined in Sec. I.

In Figs. 3(a) and 3(b) the results are summarized for the logistic map (2) without and with reset, respectively, with parameters $r=3.1$, $A_0=0.01$, and various periods $2\pi/\omega_0$. The curves of SNR vs D show clear maxima indicating the occurrence of SR. In both cases the values of SNR increase with decreasing ω_0 and the location of the maximum is shifted towards smaller values of D. Strong dependence of SNR on the periodic forcing frequency indicates an important role of dynamical effects in this model, since in nondynamical SR (e.g., in nondynamical TC systems) SNR should not vary with ω_0 [13]. The SNR reaches maximum values for $\omega_0\rightarrow 0$, i.e., in the adiabatic limit. It was found that in both cases this limit is obtained already for $2\pi/\omega_0>128$.

In the case of the logistic map with reset the SNR values are by order of magnitude greater than in the case without reset. This is because even in the case of weak noise the system (2) without reset does not follow the periodic orbits

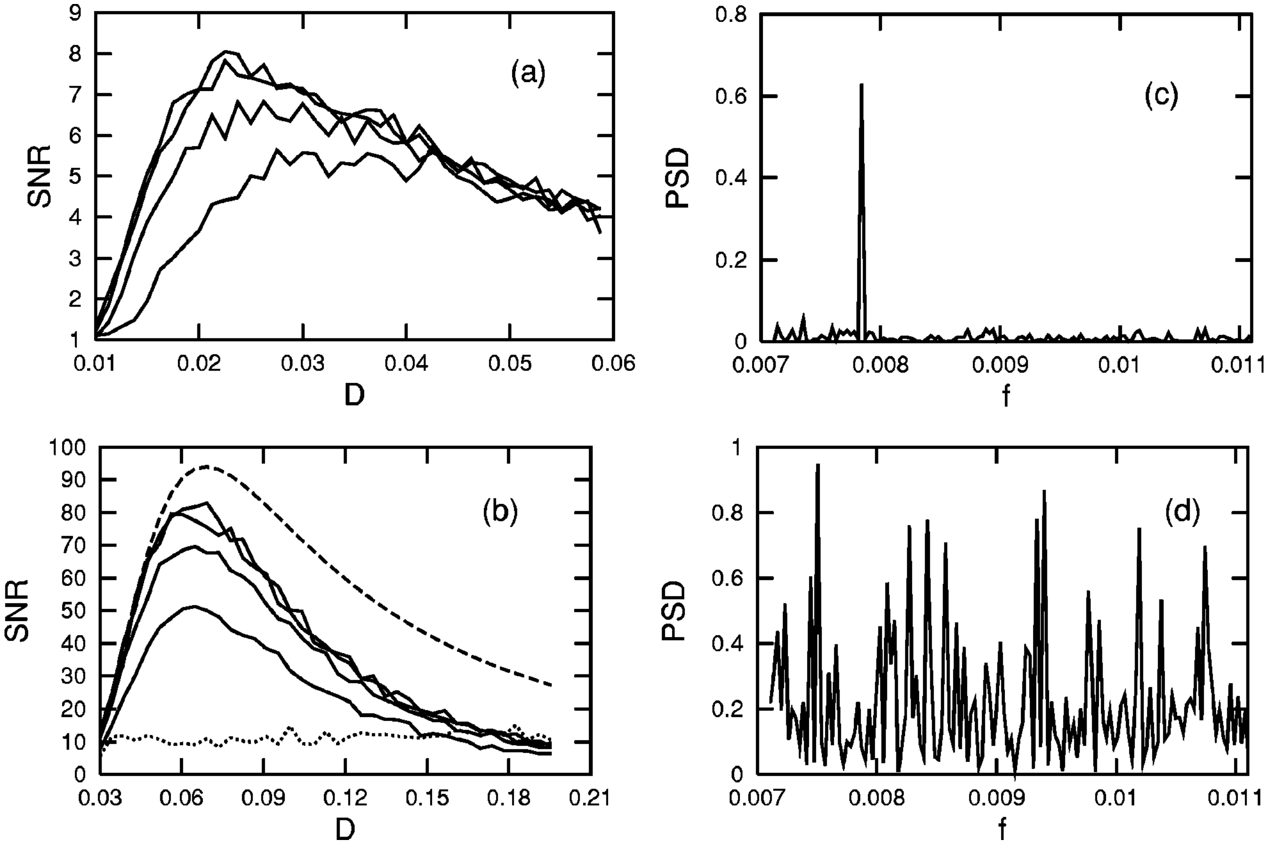

FIG. 3. (a) and (b) SNR vs D for the logistic map (2) with $r=3.1$, $A_0=0.01$ without reset (a) and with reset (b); numerical curves (solid lines) from bottom to top are for $2\pi/\omega_0=16$, 32, 64, 128. In (b) predictions of the adiabatic theory of Sec. III B (dashed line), and SNR vs B0 (multiplied by 10) for the signal z_n from the states of the second iteration of the logistic map (dotted line) are shown; (c) and (d) PSD vs f for the logistic map with reset for the signal y_n from the jumps (c), and for the signal z_n from the states of the second iteration of the map (d).

$\tilde{x}^{(1,2)}(n)$ of Eq. (2) without noise [cf. Fig. 1(b)]. The points x_n are thus not necessarily close to the separatrix $x^*(n)$ when the periodic forcing is at a maximum, and far from the separatrix when the periodic forcing is at a minimum, so the effect of periodic forcing is smaller. On the other hand, in this case the jumps can be also caused by applying small noise several times, while in the map with reset the noise has to be strong enough to force the phase jumps within one step. This effect does not compensate the decrease of SNR, but shifts the maximum of SNR towards smaller values of D. In the case of a logistic map with reset the maximum is for $D_{max}\approx 0.07$. This result can be expected since for the applied amplitude of the periodic force A_0 the minimum distance between the points $\tilde{x}^{(1,2)}(n)$ and the separatrix is of the order of 0.07, and maximum 0.14 [Fig. 1(a)], so for $D\approx D_{max}$ there will be most probably only a few phase jumps when this distance is minimum. This is an example of the cooperative effect of periodic forcing and noise, typical of SR.

In Fig. 4 SNR vs D is shown in the case of spin map (9) for $B_0=0.5$, $B_1=0.1$, and other parameters as in Sec. II B. Here the distance $B_c-B_0-B_1=0.4$ is quite large and strong noise with $D=D_{max}\approx 0.5$ is needed in order to obtain maximum SR, but the effect is clearly visible. The order of magnitude of D_{max} is again such that SR can be treated as a cooperative phenomenon resulting from periodic forcing and noise. One can hardly observe any dependence of SNR on the periodic forcing frequency for a wide range of $2\pi/\omega_0$, probably due to a large value of B_1.

In order to compare our results with the more familiar case of conventional SR in bistable systems, in Fig. 3(b) and Fig. 4 we also show SNR vs noise intensity D for the signal z_n extracted from the states rather than from jumps between the two symmetric states of the maps under study. This sig-

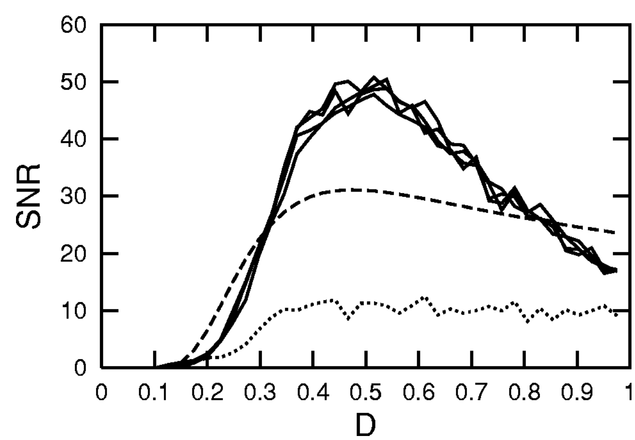

FIG. 4. SNR vs D for the spin map (9) with $\tau=2\pi$, $\lambda=0.105\,494\,2$, $A=1$, $B_0=0.5$, $B_1=0.1$; numerical curves (solid lines) for $2\pi/\omega_0=8$, 16, 128, 512 are shown (dependence of SNR on ω_0 is almost absent); the dashed curve is the result of the adiabatic theory of Sec. III B; the dotted line is SNR for the signal z_n from the states of the spin map (multiplied by 10).

nal was obtained in the following way. In the case of the logistic map with reset every second iteration was sampled, and the following signal was analyzed: $z_n=1$ if $x_n>x^*(n)$ and $z_n=-1$ if $x_n<x^*(n)$. In the case of the spin map the signal z_n was defined as $z_n=1$ if $S_{z,n}>0$ and $z_n=-1$ if $S_{z,n}<0$. This is a typical signal analyzed in the case of the conventional SR in bistable systems. In both cases the obtained in such a way values of SNR are by at least one order of magnitude smaller than in the case of SNR evaluated from the signal y_n [let us note that in Fig. 3(b) and Fig. 4 the original values of SNR are multiplied by 10] and no clear maximum of SNR can be seen. Moreover, in the case of the logistic map with reset we also show plots of the PSD vs f for the signal y_n [Fig. 3(c), the peak at the input signal frequency is clearly seen] and z_n [Fig. 3(d), only a noisy background is present in the vicinity of the input signal frequency]. These results confirm that in our systems the conventional SR effect is absent in contrast with the SR effect observed when the systems are treated as TC systems.

B. Comparison with simple adiabatic theory

In this section we compare our numerical results with predictions of a simple adiabatic theory valid for $\omega_0 \ll 1$. As pointed out in Sec. II the systems under study bear certain resemblance to TC systems. The occurrence of a jump is reflected by a peak exactly one time step long in the output signal y_n. In such systems if the probability of the occurrence of the peak $p(n)=\Pr(y_n=1)$ is a known periodic function of time with period $2\pi/\omega_0$, the SNR may be evaluated as [37,38]

$$\mathrm{SNR}=NM|P_1|^2/(\overline{p}-\overline{p^2}), \tag{11}$$

where $N=2\pi/\omega_0$, M is the number of periods within the measured time interval (thus $MN=32\,678$ is the length of the interval from which the data were stored), P_1 is the first Fourier coefficient of $p(n)$ evaluated as $P_1=(1/N)\sum_{j=0}^{N-1}p(j)\exp(-i\omega_0 j)$, and the bar denotes the time average. Equation (11) is particularly useful in the case of discrete-time systems [37,38]; the value of SNR in Eq. (11) is evaluated for a bandwidth $\Delta f=1/NM=2^{-15}$ Hz which enables a direct comparison with the SNR obtained numerically in Sec. III A.

In the case of a logistic map with reset $p(n)$ can be evaluated as follows. Assuming that the deterministic part of Eq. (2) maps the point $x^{(1)}(n-1)$ into $x^{(2)}(n)$ and vice versa [where $x^{(1)}(n-1)$, $x^{(2)}(n)$ are given by Eq. (3)] we can write the conditional probabilities that at time step n the phase jump occurred provided that $x_{n-1}=x^{(i)}(n-1)$, $i=1,2$, as

$$\begin{aligned}
&\Pr[y_n=1|x_{n-1}=x^{(1)}(n-1)]\\
&\quad=\frac{1}{\sqrt{2\pi}D}\int_{-\infty}^{x^*(n)}\exp\left\{-\frac{[\xi-x^{(2)}(n)]^2}{2D^2}\right\}d\xi,\\
&\Pr[y_n=1|x_{n-1}=x^{(2)}(n-1)]\\
&\quad=\frac{1}{\sqrt{2\pi}D}\int_{x^*(n)}^{\infty}\exp\left\{-\frac{[\xi-x^{(1)}(n)]^2}{2D^2}\right\}d\xi.
\end{aligned} \tag{12}$$

Denoting by $\Pr[x_n=x^{(i)}(n)]$ the probability of occupation of $x^{(i)}(n)$ at time step n, the total probability of the jump at time n is

$$\begin{aligned}
p(n)&=\Pr(y_n=1)\\
&=\Pr[y_n=1|x_{n-1}=x^{(1)}(n-1)]\\
&\quad\times\Pr[x_{n-1}=x^{(1)}(n-1)]\\
&\quad+\Pr[y_n=1|x_{n-1}=x^{(2)}(n-1)]\\
&\quad\times\Pr[x_{n-1}=x^{(2)}(n-1)].
\end{aligned} \tag{13}$$

In the first approximation the probabilities of occupation of the stable fixed points in Eq. (13) can be assumed as 1/2. However, the location of the stable fixed points is not symmetric with respect to the separatrix. Thus improved values of the probabilities can be evaluated from the Chapman-Kolmogorov equation for $\Pr[x_n=x^{(i)}(n)]$, $i=1,2$. After simple algebra we get

$$\begin{aligned}
&\Pr[x_n=x^{(1)}(n)]\\
&\quad=\Pr[y_n=1|x_{n-1}=x^{(1)}(n-1)]\Pr[x_n=x^{(1)}(n)]\\
&\qquad+\{1-\Pr[y_n=1|x_{n-1}=x^{(2)}(n-1)]\}\\
&\qquad\times\Pr[x_n=x^{(2)}(n)],\\
&\Pr[x_n=x^{(1)}(n)]+\Pr[x_n=x^{(2)}(n)]=1.
\end{aligned} \tag{14}$$

The above equations were obtained under the assumption that the probabilities of occupation of any fixed point at time steps $n-1$ and n are equal, which is true for $\omega_0 \ll 1$, and only the conditional probabilities of phase jumps vary with n.

Using Eqs. (12)–(14) we evaluated $p(n)$ and calculated SNR from Eq. (11). The results are shown in Fig. 3(b). The theoretical curve lies above the numerical one, in particular for high noise intensities. This discrepancy can be attributed to the above-mentioned asymmetry of points $x^{(1)}(n)$ and $x^{(2)}(n)$ with respect to the separatrix $x^*(n)$. Equation (11) is exact in *nondynamical* TC systems, in which the periodic, or deterministic, and stochastic components of the system can be separated [37]. In the logistic map with reset the deterministic component is a superposition of the overall period-2 pattern and the effect of additive periodic forcing. Let us assume that at $n=0$ we have $x_0=x^{(1)}(0)$. Then, depending on the number of phase jumps within one period, at $n=N=2\pi/\omega_0$, after reset, we can have $x_N=x^{(1)}(0)$, but also $x_N=x^{(2)}(0)$. Probabilities of the phase jump in the iteration $N+1$ are different in these two cases. Thus the addition of random white noise in Eq. (2) introduces also a certain degree of randomness into the deterministic part of the dynamics of the logistic map with reset, and the deterministic and stochastic components cannot be separated. In our theory this effect is lost since $p(n)$ is obtained from Eq. (13) by averaging the conditional probabilities of the phase jump $\Pr[y_n=1|x_{n-1}=x^{(1)}(n-1)]$ and $\Pr[y_n=1|x_{n-1}=x^{(2)}(n-1)]$, but it can lead to the decrease of SNR.

In the case of the spin map (9) a quantitative theory can be based on the expression for the mean time τ between jumps between two symmetric parts of the attractor above the attractor merging crisis [34]. For the parameters used in this paper it was found that for constant B this time scales as $\tau=\alpha(B-B_c)^{-\gamma}$ with $\alpha\approx2.0$ and $\gamma\approx0.77$ [27–29], and the probability of the jump is $1/\tau$. In the case of noise-free SR adiabatic theories based on such scaling were used in [15,16,20]. In the presence of the periodic forcing and noise in Eq. (10) the probability of the jump can be evaluated as the average

$$p(n)=\Pr(y_n=1)$$
$$=\frac{1}{\sqrt{2\pi}\alpha D}\int_{B_c-B_0-B_1\cos\omega_0 n}^{\infty}(B_0+B_1\cos\omega_0 n+\xi-B_c)^{\gamma}\times\exp\left(-\frac{\xi^2}{2D^2}\right)d\xi \quad (15)$$

and SNR can be again evaluated from Eq. (11). Comparison of the theoretical prediction with numerical results (Fig. 4) shows that in this case the agreement is much worse than in the case of the logistic map. First, this can be caused by the fact that Eq. (11), derived for nondynamical systems, neglects any complicated internal system dynamics. Second, one should remember that the scaling relation for τ on which Eq. (15) is based is valid only for B very close to B_c, and in the present case the noise component is so strong that the values of B in Eq. (10) can much exceed B_c. Third, Eq. (15) is only approximate and it is not necessarily true that the probability of the jump in the presence of noise can be obtained by averaging the probabilities in the absence of noise [36].

It should be noted that in both cases of Fig. 3(b) and Fig. 4 the location of maxima of the curves SNR vs D agrees quite well with the numerical results. Hence the simple adiabatic theories discussed here can be used at least to fit the noise intensity for the maximum SR efficiency.

IV. SUMMARY AND CONCLUSIONS

In this paper we investigated SR in discrete-time dynamical systems: the logistic map and the spin map with periodic forcing and noise. The systems under study model dynamical TC systems driven by the subthreshold periodic signal which is too weak to cause the system to cross the assumed threshold. In such systems, in the presence of noise the probability of the occurrence of certain events like jumps or pulses of unit length, which indicate TC, is modulated periodically so that it reaches its maximum value once per period. If the noise strength is chosen properly, the maximum value of SNR in the time series reflecting the occurrence of TC events is expected. In this paper both systems under study are bistable, since they possess two distinct symmetric states (phase-shifted period-2 orbits or disjoint attractors before crisis), but this bistability plays a quite different role as compared to conventional SR. The important fact is that jumps between these two symmetric states can be modeled as TC events. It follows that the presence or absence of SR in systems studied in our paper depends not just on the systems considered but on the kind of observed signal as well. In both systems SR was a dynamical effect. This is indicated by the dependence of SNR on ω_0. This distinguishes our models from the case of nondynamical SR in TC systems [13] which can also easily be described in terms of discrete-time systems [37,38].

The logistic map (2) with the parameters used in this paper is a discrete-time model of systems close to the period-doubling bifurcation, moving on the period-2 orbit modulated by periodic forcing. In this model we showed the possibility to obtain SR using additive noise which induces phase jumps in the overall period-2 pattern. The SNR at the periodic forcing frequency from the signal indicating the occurrence of phase jumps can be maximized by the optimum choice of the noise strength. Two cases were discussed: in the first case the initial condition for the $n+1$ step was just the value of x_n resulting from Eq. (2), and in the second one, called the logistic map with reset, the initial condition for the $n+1$ step was chosen always on the periodically modulated period-2 orbit of the noise-free system. One should remember that discrete-time maps are usually models resulting from stroboscopic sampling of continuous-time systems dynamics. Thus the model with reset can be interpreted as a discrete-time model of continuous-time systems driven by pulses of external noise, which evolve freely between consecutive pulses. If in such systems the period-2 orbit is strongly attracting and if the pulses are applied only at certain moments of time, e.g., when crossing a Poincaré plane, then just before the pulse the system will already be on the orbit resulting from the deterministic dynamics. The occurrence of the phase jump means then the shift occurs in the exact period-2 sequence of these crossings. The map without reset can in turn be interpreted as a model of continuous-time systems driven by incessant noise. In this case the location of points on the Poincaré section will always be somewhat random, and the occurrence of the phase jump means only the perturbation of the overall period-2 pattern, as discussed in Sec. II A. This greater degree of randomness leads to the decrease of the effect of periodic forcing on the system dynamics, as discussed in Sec. III A, and to the decrease of SNR, although the maximum of SNR occurs for smaller noise strength.

The spin map investigated in this paper forms a discrete-time model of SR in dynamical systems with an attractor merging and multiplicative periodic forcing and noise. We chose the parameters so that for $B_0-B_1<B<B_0+B_1$ in Eq. (10) the two symmetric attractors are stable fixed points and there is no possibility of jumps between them. The jumps appear only in the presence of noise, due to the occurrence of the noise-induced attractor merging crisis. Neglecting the possible chaotic dynamics for B slightly below B_c, the system under study is a model of a symmetric bistable system with the barrier height modulated by periodic forcing and noise.

The examples considered in this paper represent an extension of the idea of using simple discrete-time systems to model SR in more complicated continuous-time systems to the case of TC systems. It is shown that this kind of SR can be realized in systems close to bifurcation points or crises. The maps considered here have many properties generic for chaotic systems. Hence our maps can model many other experimental chaotic systems with period doubling and crises in which SR can be observed.

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